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The Six-Vertex Model Eigenvectors as Critical Limit of the Eight-Vertex Model Bethe Ansatz

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The critical limit of the eight-vertex model eigenvectors obtained by means of the generalized Bethe Ansatz is shown to give the six-vertex eigenvectors as constructed in a previous paper by two of the authors. Furthermore, an explicit mapping is established between these eigenvectors and the usual Bethe Ansatz eigenvectors of the six-vertex model. This allows one to show that the index v labeling the eight-vertex eigenstates becomes exactly the third component of the total spin in the critical limit.

KEY WORDS: Exactly solved models; two-dimensional vertex models; Bethe Ansatz.

1. INTRODUCTION

The Bethe Ansatz provides one of the nicest and most effective ways to exactly solve an eigenvalue problem in physics. In statistical mechanics, the pioneering works on six-[(1)] and eight-vertex[(2,3)] models (with two states per bond) and their more recent generalizations to a large variety of multistate vertex models (with $q \geqslant 2$ states per bond) (see, e.g., ref. 4) represent the highlights of the Bethe Ansatz technique.

The six-vertex model is actually a limit case of the eight-vertex model. However, the Bethe Ansatz solution of the latter in refs. 2 and 3 is much more involved than its six-vertex counterpart. Moreover, when the limit that leads to the six-vertex model is taken in the results of refs. 2 and 3, singularities appear in different steps of the calculations. As a consequence, no systematic study has been carried out, to our knowledge, on the connec-

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tion between the usual Bethe Ansatz for the six-vertex model and the critical limit of the generalized Bethe Ansatz introduced in refs. 2 and 3.

In a recent paper[5] an alternative Bethe Ansatz for the six-vertex model has been developed. It is based on the methods applied in ref. 3 to the eight-vertex case and is well suited to study the connection with critical RSOS models.[6]

In the present paper we investigate and clarify the connection between the eight-vertex eigenvectors and those of the six-vertex in their two available constructions.[1,4,5] Our results can be summarized as follows:

1. The eight-vertex eigenvectors, in an appropriate parametrization which eliminates all divergences, become the six-vertex eigenvectors of ref. 5 in the critical limit.

2. There exists an explicit operator mapping these vectors into those of the usual six-vertex Bethe Ansatz [Eqs. (3.19)–(3.21), (3.26), and (3.27)].[4]

3. The set of Bethe Ansatz equations of ref. 5 can be transformed in a natural way into those of the conventional Bethe Ansatz.

From these results it appears evident that the integer p of ref. 5, which naturally represents the quantum number associated with a conserved quantity, is identical to the third component S_3 of the total spin of the quantum XXZ chain associated with the transfer matrix of the six-vertex model. Hence the eight-vertex index ν of ref. 3 just becomes, in the critical limit, the eigenvalue of S_3.

In the Appendix we analyze in detail the case of the transfer matrix corresponding to four sites. By an explicit calculation, we show that in this case the two sets of eigenvectors corresponding to the two alternative Bethe Ansatz are in fact identical (up to an irrelevant multiplication factor). This is indeed the only possibility, because there is no accidental degeneracy in the spectrum of the four-site, six-vertex transfer matrix.

2. THE EIGENVECTORS OF THE EIGHT-VERTEX MODEL AND THEIR CRITICAL LIMIT

In this section we briefly review the Bethe-Ansatz construction of the eigenvalues and eigenvectors of the eight-vertex transfer matrix, closely following refs. 2 and 3.

[4] However, due to the complicated nonlocal structure of this operator, the complete equivalence of the two sets of eigenvectors could not be proved in general. This problem is intimately related to the accidental degeneracies of the transfer matrix, about which almost nothing is known.

The R-matrix of the model reads

$$R(\theta)=\begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} \tag{2.1}$$

where the weights a, b, c, and d are parametrized as

$$\begin{aligned} a(\theta, \gamma, k) &= \Theta(\theta)\, \Theta(\gamma)\, H(\theta+\gamma) \\ b(\theta, \gamma, k) &= \Theta(\theta)\, H(\gamma)\, \Theta(\theta+\gamma) \\ c(\theta, \gamma, k) &= \Theta(\gamma)\, H(\theta)\, \Theta(\theta+\gamma) \\ d(\theta, \gamma, k) &= H(\theta)\, H(\gamma)\, H(\theta+\gamma) \end{aligned} \tag{2.2}$$

Here, $\Theta(z)$ and $H(z)$ are Jacobi theta functions of modulus k, θ is the spectral parameter, and γ is the anisotropy. The six-vertex model follows in the limit $k \to 0$, since then $\Theta(z) \to 0$, while $H(z)/\sqrt{k} \to \sin z$, and the R-matrix (2.1), once divided by $\sqrt{k}$, reduces to that of the six-vertex model.

The monodromy matrix, which is the fundamental object in the algebraic formulation of the Bethe Ansatz,[3] reads

$$T_{ab}(\theta)_{\boldsymbol{\alpha\beta}} = \sum_{a_1,\ldots,a_{N-1}} t_{aa_1}(\theta)_{\alpha_1\beta_1} t_{a_1a_2}(\theta)_{\alpha_2\beta_2} \cdots t_{a_{N-1}b}(\theta)_{\alpha_N\beta_N} \tag{2.3}$$

where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_N)$, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_N)$, and every index takes two values. The integer N is the number of sites in the horizontal direction of a square lattice of size $N \times M$, and the (normalized) local operators $t_{ab}(\theta)$ are given in term of the R-matrix by

$$t_{ab}(\theta)_{\alpha\beta} = \frac{1}{H(\theta+\gamma/2)}\, R\left(\theta - \frac{\gamma}{2}\right)_{\alpha a, b\beta} \tag{2.4}$$

The monodromy matrix (2.3) is a two-by-two matrix with operator entries and is usually written as

$$T(\theta) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \tag{2.5}$$

Its fundamental property is to satisfy the Yang–Baxter algebra

$$R(\theta-\theta')[T(\theta)\otimes T(\theta')] = [T(\theta')\otimes T(\theta)]\, R(\theta-\theta') \tag{2.6}$$

implying that the transfer matrix for periodic boundary conditions, $\tau(\theta) = A(\theta) + D(\theta)$, belongs to a commuting family

$$[\tau(\theta), \tau(\theta')] = 0 \tag{2.7}$$

and is therefore diagonalized by θ-independent eigenvectors.

The construction of these eigenvectors proceeds now from (2.6) in a completely algebraic way.[3] One introduces the "intertwining" vectors

$$X_n(\theta) = \begin{pmatrix} H(z_n + s - \theta) \\ \Theta(z_n + s - \theta) \end{pmatrix}, \qquad Y_n(\theta) = \frac{1}{h(\tau_{2n})} \begin{pmatrix} H(z_n + t + \theta) \\ \Theta(z_n + t + \theta) \end{pmatrix} \tag{2.8}$$

where

$$z_n = n\gamma + (\pi - iK')/2, \qquad h(z) = \Theta(0)\, \Theta(z)\, H(z)$$
$$\tau_n = -K + (n\gamma + s + t + \pi - iK')/2$$

In all the above relations K and iK' represent the quarter periods of the elliptic functions $H(z)$ and $\Theta(z)$, n is an integer, while s and t are arbitrary complex parameters. They are related by the k-dependent shift $s \to s + (\pi - iK')/2$, $t \to t + (\pi - iK')/2$ to those introduced in ref. 3. With our choice for s and t the limit $k \to 0$ that leads to the six-vertex model turns out to be well defined. The fundamental defining properties of the intertwining vectors X_n and Y_n are

$$\begin{aligned} R(\theta - \theta')&[X_n(\theta) \otimes X_{n+1}(\theta')] \\ &= h(\theta - \theta' + \gamma)[X_n(\theta') \otimes X_{n+1}(\theta)] \end{aligned} \tag{2.9}$$

$$\begin{aligned} R(\theta - \theta')&[Y_{n+1}(\theta) \otimes Y_n(\theta')] \\ &= h(\theta - \theta' + \gamma)[Y_{n+1}(\theta') \otimes Y_n(\theta)] \end{aligned} \tag{2.10}$$

$$\begin{aligned} R(\theta - \theta')&[Y_m(\theta) \otimes X_n(\theta')] \\ &= \frac{h(\gamma)\, h(\tau_{m+n+1} + \theta - \theta')}{h(\tau_{m+n+1})} [Y_m(\theta') \otimes X_n(\theta)] \\ &\quad + \frac{h(\theta - \theta')\, h(\tau_{m+n-1})\, h(\tau_{2m+2})}{h(\tau_{m+n+1})\, h(\tau_{2m})} [X_{n+1}(\theta') \otimes Y_{m+1}(\theta)] \end{aligned}$$

$$\begin{aligned} R(\theta - \theta')&[X_m(\theta) \otimes Y_n(\theta')] \\ &= \frac{h(\gamma)\, h(\tau_{m+n-1} - \theta + \theta')}{h(\tau_{m+n-1})} [X_m(\theta') \otimes Y_n(\theta)] \\ &\quad + \frac{h(\theta - \theta')\, h(\tau_{m+n+1})\, h(\tau_{2m-2})}{h(\tau_{m+n-1})\, h(\tau_{2m})} [Y_{n-1}(\theta') \otimes X_{m-1}(\theta)] \end{aligned}$$

It follows from these equations and from the Yang–Baxter algebra (2.6) that, if $M_n(\theta)$ is the two-by-two matrix $(X_n(\theta)\ Y_n(\theta))$, then the elements of the gauge-transformed monodromy matrix

$$T_{mn}(\theta)=M_m(\theta)^{-1}\,T(\theta)\,M_n(\theta)=\begin{pmatrix}A_{mn}(\theta) & B_{mn}(\theta)\\ C_{mn}(\theta) & D_{mn}(\theta)\end{pmatrix} \tag{2.11}$$

satisfy the following commutation relations:

$$B_{m,n+1}(\theta)\,B_{m+1,n}(\theta')=B_{m,n+1}(\theta')\,B_{m+1,n}(\theta) \tag{2.12}$$

$$\begin{aligned}A_{mn}(\theta)\,B_{m+1,n-1}(\theta')&=\alpha(\theta-\theta')\,B_{m,n-2}(\theta')\,A_{m+1,n-1}(\theta)\\&\quad-\beta_{n-1}(\theta-\theta')\,B_{m,n-2}(\theta)\,A_{m+1,n-1}(\theta')\end{aligned} \tag{2.13}$$

$$\begin{aligned}D_{mn}(\theta)\,B_{m+1,n-1}(\theta')&=\alpha(\theta-\theta')\,B_{m+2,n}(\theta')\,D_{m+1,n-1}(\theta)\\&\quad+\beta_{n+1}(\theta-\theta')\,B_{m+2,n}(\theta)\,D_{m+1,n-1}(\theta')\end{aligned} \tag{2.14}$$

where

$$\alpha(\theta)=\frac{h(\theta-\gamma)}{h(\theta)},\qquad \beta_n(\theta)=\frac{h(\gamma)\,h(\theta-\tau_n)}{h(\theta)\,h(\tau_n)}$$

Now consider the vectors

$$\Psi_n(\boldsymbol{\theta})=B_{n+1,n-1}(\theta_1)\,B_{n+2,n-2}(\theta_2)\cdots B_{n+r,n-r}(\theta_r)\,\Omega_n^{(N)} \tag{2.15}$$

where $\Omega_n^{(N)}$ is the direct-product state

$$\Omega_n^{(N)}=X_{n+1}(\gamma/2)\otimes X_{n+2}(\gamma/2)\otimes\cdots\otimes X_{n+N}(\gamma/2) \tag{2.16}$$

and $\boldsymbol{\theta}=(\theta_1,\dots,\theta_r)$ are, for the moment, arbitrary parameters.[5] Using the commutation rules (2.12)–(2.14) and the properties of $\Omega_n^{(N)}$,

$$\begin{aligned}A_{N+n,n}(\theta)\,\Omega_n^{(N)}&=\left[\frac{h(\theta+\gamma/2)}{H(\theta+\gamma/2)}\right]^N\Omega_{n-1}^{(N)}\\D_{N+n,n}(\theta)\,\Omega_n^{(N)}&=\left[\frac{h(\theta-\gamma/2)}{H(\theta+\gamma/2)}\right]^N\Omega_{n+1}^{(N)}\\C_{N+n,n}(\theta)\,\Omega_n^{(N)}&=0\end{aligned} \tag{2.17}$$

[5] Due to the double periodicity of all elliptic functions involved, one may constrain $\theta_1,\dots,\theta_r$ to lie in the fundamental rectangle. Moreover, due to Eq. (2.14), the ordering of the θ_j is irrelevant.

one can show[3] that the Fourier-transformed states

$$\Psi_\phi(\boldsymbol{\theta}) = \sum_{n=-\infty}^{+\infty} e^{in\phi} \Psi_n(\boldsymbol{\theta}) \tag{2.18}$$

are eigenvectors of the transfer matrix $\tau(\theta) = A(\theta) + D(\theta) = A_{nn}(\theta) + D_{nn}(\theta)$ according to

$$\begin{aligned} \tau(\theta)\, \Psi_\phi(\boldsymbol{\theta}) &= [e^{i\phi} \Lambda_+(\theta, \boldsymbol{\theta}) + e^{-i\phi} \Lambda_-(\theta, \boldsymbol{\theta})]\, \Psi_\phi(\boldsymbol{\theta}) \\ \Lambda_\pm(\theta, \boldsymbol{\theta}) &= \left[\frac{h(\theta \pm \gamma/2)}{H(\theta + \gamma/2)}\right]^N \prod_{l=1}^{r} \frac{h(\theta - \theta_l \mp \gamma)}{h(\theta - \theta_l)} \end{aligned} \tag{2.19}$$

provided the parameters $\boldsymbol{\theta}$ satisfy the so-called Bethe Ansatz equations:

$$\left[\frac{h(\theta_l + \gamma/2)}{h(\theta_l - \gamma/2)}\right]^N = -e^{-2i\phi} \prod_{k=1}^{r} \frac{h(\theta_l - \theta_k + \gamma)}{h(\theta_l - \theta_k - \gamma)} \tag{2.20}$$

One can also show that if γ is not a linear combination of $4K$ and $2iK'$ with integer coefficients, then the natural number r in the above equations must be equal to $N/2$. Moreover, the state (2.18) will vanish unless the angle parameter ϕ takes some special values. It is argued in ref. 3 that these are given by

$$\phi = \phi_p = \frac{\pi\gamma}{2K} p \tag{2.21}$$

where p is an arbitrary integer (it is denoted by v in ref. 3). In the particular case that $\gamma = 4mK/Q$, with m and Q positive integers, then the condition $2r = N$ can be relaxed to $2r = N \pmod{Q}$ and consequently the sum over n in (2.18) can be restricted to the range $0, 1, \ldots, Q-1$.

Let us now consider the $k \to 0$ (six-vertex) limit. As mentioned above, in this limit the R-matrix (2.1) essentially becomes that of the six-vertex model, since, up to an overall factor $\sqrt{k}$, $a \to \sin(\theta + \gamma)$, $b \to \sin\gamma$, $c \to \sin\theta$, and $d \to 0$. In particular, if θ and γ are real, we are dealing with the antiferroelectric gapless regime of the six-vertex model.[1] We then observe that, with our parametrizations (2.2) and (2.8) of the R-matrix elements and intertwining vectors, resp., and with the choice (2.16) for the "quasi-refence state," the limit $k \to 0$ can be taken smoothly in all the formulas above. In particular, the intertwining vectors become

$$X_n(\theta) = \begin{pmatrix} e^{i(n\gamma - \theta + s)} \\ 1 \end{pmatrix}, \qquad Y_n(\theta) = \begin{pmatrix} e^{i\theta + i(t - s - \pi)/2} \\ e^{-in\gamma - i(t + s - \pi)/2} \end{pmatrix} \tag{2.22}$$

and are compatible with those introduced in ref. 5 directly for the six-vertex model. It is then easy to check that the eight-vertex eigenvectors (2.15) reduce to the six-vertex eigenstates as constructed in ref. 5. As for the transfer matrix eigenvalues [Eq. (2.19)] and the Bethe Ansatz equations (2.20), the $k \to 0$ limit gives

$$\Lambda_{\pm}(\theta, \boldsymbol{\theta}) = \left[\frac{\sin(\theta \pm \gamma/2)}{\sin(\theta + \gamma/2)}\right]^N \prod_{l=1}^{r} \frac{\sin(\theta - \theta_l \mp \gamma)}{\sin(\theta - \theta_l)} \tag{2.23}$$

$$\left[\frac{\sin(\theta_l + \gamma/2)}{\sin(\theta_l - \gamma/2)}\right]^N = -e^{-2i\phi} \prod_{k=1}^{r} \frac{\sin(\theta_l - \theta_k + \gamma)}{\sin(\theta_l - \theta_k - \gamma)} \tag{2.24}$$

where now $\phi_p = p\gamma$ [cf. (2.21)]. As expected, Eqs. (2.23) and (2.24) coincide with those found in ref. 5. This concludes the proof that the "generalized" Bethe Ansatz solution of the six-vertex model[5] is the smooth $k \to 0$ limit of the (suitably normalized) generalized Bethe Ansatz of the eight-vertex model.[3]

3. THE EIGENVECTORS OF THE SIX-VERTEX MODEL: EQUIVALENCE BETWEEN THE TWO BETHE ANSATZE

In this section we will study the relationship between the six-vertex eigenvectors constructed in ref. 5 and rederived in the previous section as critical limit of the eight-vertex case, and those known from earlier works.[1,3,4] Let us first summarize the latter well-known construction.

Consider the R-matrix [Eq. (2.1)], the local operators $t_{ab}(\theta)$ [Eq. (2.4)], and the monodromy matrix $T_{ab}(\theta)$ [Eqs. (2.3)–(2.5)] for the six-vertex model. This means that $a = \sin(\theta + \gamma)$, $b = \sin\gamma$, $c = \sin\theta$, and $d = 0$, rather than as in (2.2), and that $H(\theta + \gamma/2)$ is to be replaced by $\sin(\theta + \gamma/2)$ in (2.4). Hence, from now on,

$$R(\theta) = \begin{pmatrix} \sin(\theta+\gamma) & 0 & 0 & 0 \\ 0 & \sin(\gamma) & \sin(\theta) & 0 \\ 0 & \sin(\theta) & \sin(\gamma) & 0 \\ 0 & 0 & 0 & \sin(\theta+\gamma) \end{pmatrix} \tag{3.1}$$

$$t_{ab}(\theta)_{\alpha\beta} = \frac{1}{\sin(\theta + \gamma/2)} R\left(\theta - \frac{\gamma}{2}\right)_{\alpha a, b\beta} \tag{3.2}$$

The block-diagonal form of $R(\theta)$ allows now for the existence of a local reference state $|+\rangle$ annihilated by $t_{21}(\theta)$ and eigenstate of both $t_{11}(\theta)$ and $t_{22}(\theta)$. Indeed, simply setting $|+\rangle = \binom{1}{0}$ and $|-\rangle = \binom{0}{1}$, it is easy to check that

$$t(\theta)\,|+\rangle = \begin{pmatrix} t_{11}\,|+\rangle & t_{12}\,|+\rangle \\ t_{21}\,|+\rangle & t_{22}\,|+\rangle \end{pmatrix} = \begin{pmatrix} |+\rangle & \hat{b}\,|-\rangle \\ 0 & \hat{c}\,|+\rangle \end{pmatrix} \tag{3.3}$$

where $\hat{b} = \sin\gamma/\sin(\theta+\gamma/2)$ and $\hat{c} = \sin(\theta-\gamma/2)/\sin(\theta+\gamma/2)$. It follows from (3.3) that the "ferromagnetic" global state

$$|N/2\rangle = |+ + \cdots +\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

acts as reference state for the monodromy matrix $T(\theta)$:

$$\begin{aligned} T(\theta)\,|N/2\rangle &= \begin{pmatrix} A(\theta)\,|N/2\rangle & B(\theta)\,|N/2\rangle \\ C(\theta)\,|N/2\rangle & D(\theta)\,|N/2\rangle \end{pmatrix} \\ &= \begin{pmatrix} |N/2\rangle & B(\theta)\,|N/2\rangle \\ 0 & \hat{c}^N\,|N/2\rangle \end{pmatrix} \end{aligned} \tag{3.4}$$

and is clearly an eigenstate of the transfer matrix $\tau(\theta) = A(\theta) + D(\theta)$ with eigenvalue $1+\hat{c}^N$. According to the (algebrized) Bethe Ansatz, all the eigenvectors of $\tau(\theta)$ are of the form

$$\Phi(\boldsymbol{\zeta}) = B(\zeta_1)\,B(\zeta_2)\cdots B(\zeta_q)\,|N/2\rangle \tag{3.5}$$

where the parameters $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_q)$, $1 \leqslant q \leqslant N/2$, satisfy the equations

$$\left[\frac{\sin(\zeta_l+\gamma/2)}{\sin(\zeta_l-\gamma/2)}\right]^N = \prod_{k=1}^{q} \frac{\sin(\zeta_l-\zeta_k+\gamma)}{\sin(\zeta_l-\zeta_k-\gamma)} \tag{3.6}$$

Notice that, as compared with the other generalized Bethe Ansatz equations (2.24), Eqs. (3.6) differ in two respects: there is no phase factor $e^{-2i\phi}$ and the integer q, unlike r in (2.24), can take all values between 0 and $N/2$. For this reason we have preferred to use different symbols, resp. $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$, for the roots of the two sets of equations, which are different in general (unless, e.g., $\phi = 0$ and $q = N/2$). The phase factor is absent also in the expression for the $\tau(\theta)$ eigenvalues of $\Phi(\boldsymbol{\theta})$, which now reads [cf. (2.19)]

$$\tau(\theta)\,\Phi(\boldsymbol{\zeta}) = [\Lambda_+(\theta,\boldsymbol{\zeta}) + \Lambda_-(\theta,\boldsymbol{\zeta})]\,\Phi(\boldsymbol{\zeta}) \tag{3.7}$$

with the functional form of $\Lambda_\pm$ given by (2.23).

The Bethe Ansatz (3.5) is directly based on the commutation rules for the operators A, B, C, and $D(\theta)$ determined by the Yang–Baxter algebra (2.6), with no need to introduce intertwining vectors for the R-matrix. In particular, the algebra (2.6) implies that $[B(\theta), B(\theta')] = 0$. Moreover, due to the $U(1)$ invariance of the six-vertex model (σ_j, $j = 1, 2, 3$, are Pauli matrices)

$$[R(\theta), e^{i\lambda\sigma_3} \otimes e^{i\lambda\sigma_3}] = 0 \tag{3.8}$$

the operators $B(\theta)$ act as lowering step operators with respect to the total third component of the spin[6]

$$[S_3, B(\theta)] = -B(\theta), \qquad S_3 = \frac{1}{2} \sum_{n=1}^{N} \sigma_3^{(n)} \tag{3.9}$$

It follows that the states (3.5), unlike the ($k \to 0$ limit) of the generalized states $\Psi_n(\boldsymbol{\theta})$, have a definite S_3 eigenvalue

$$S_3 \Phi(\zeta) = (N/2 - q)\, \Phi(\zeta) \tag{3.10}$$

It is natural to search for a relationship between the two alternative Bethe Ansatz constructions briefly described up to now. This task is most easily accomplished by graphical methods. Suppose we depict the local operators $t_{ab}(\theta)$ as in Fig. 1. Then the states $\Phi(\zeta)$ have the graphical representation of Fig. 2. Next consider the defining relations (2.9)–(2.12) for the six-vertex model (that is, take $k \to 0$ and drop the overall factor $\sqrt{k}$). They read

$$\begin{aligned} R(\theta-\theta')&[X_n(\theta) \otimes X_{n+1}(\theta')] \\ &= \sin(\theta-\theta'+\gamma)[X_n(\theta') \otimes X_{n+1}(\theta)] \end{aligned} \tag{3.11}$$

$$\begin{aligned} R(\theta-\theta')&[Y_{n+1}(\theta) \otimes Y_n(\theta')] \\ &= \sin(\theta-\theta'+\gamma)[Y_{n+1}(\theta') \otimes Y_n(\theta)] \end{aligned} \tag{3.12}$$

$$\begin{aligned} R(\theta-\theta')&[Y_m(\theta) \otimes X_n(\theta')] \\ &= \sin(\gamma)[Y_m(\theta') \otimes X_n(\theta)] \\ &\quad + \sin(\theta-\theta')[X_{n+1}(\theta') \otimes Y_{m+1}(\theta)] \end{aligned} \tag{3.13}$$

[6] Henceforth we adopt the standard notation

$$A^{(n)} = \overbrace{I \otimes \cdots \otimes I}^{n-1 \text{ times}} \otimes A \otimes \overbrace{I \otimes \cdots \otimes I}^{N-n \text{ times}}$$

where A is any one-site operator (i.e., a two-by-two matrix) and I is the two-by-two identity.

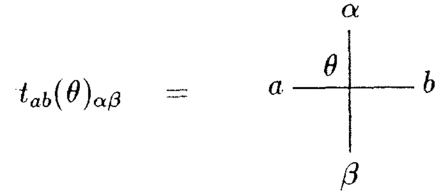

Fig. 1. Graphical representation of the matrix elements of the local operator $t_{ab}(\theta)$.

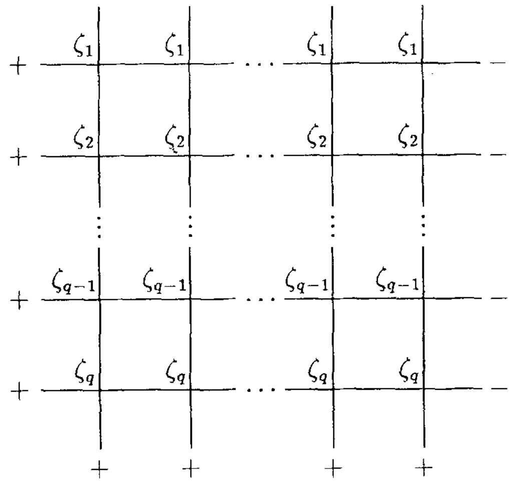

Fig. 2. The "conventional" Bethe Ansatz states $B(\zeta_1)\,B(\zeta_2)\cdots B(\zeta_q)\,|N/2\rangle$: each row represent the application of one global operator $B(\zeta_j)$.

$$\begin{aligned} R(\theta-\theta')[X_m(\theta)\otimes Y_n(\theta')] \\ = \sin(\gamma)[X_m(\theta')\otimes Y_n(\theta)] \\ + \sin(\theta-\theta')[Y_{n-1}(\theta')\otimes X_{m-1}(\theta)] \end{aligned} \tag{3.14}$$

Let us observe that also $|\pm\rangle$ can trivially act as intertwining vectors for the six-vertex R-matrix. In fact, by the definition itself of $R(\theta)$ as a four-by-four matrix [see (3.1)], Eqs. (3.11)–(3.14) hold also after the substitution

$$X_n(\theta)\to|+\rangle, \qquad Y_n(\theta)\to|-\rangle \tag{3.15}$$

Graphically, when $\theta'=\gamma/2$, equations like (3.11)–(3.14) can be depicted as in Fig. 3 [recall Eq. (3.2)]. Noice also that the intertwining vectors for the six-vertex model, generically defined as solutions of Eqs. (3.11)–(3.14), can be written in a more symmetric and general form as [compare Eq. (2.22)]

$$X_n(\theta)=\begin{pmatrix} x_+\,e^{-i(\theta-n\gamma)/2} \\ x_-\,e^{-i(\theta-n\gamma)/2}\end{pmatrix}, \qquad Y_n(\theta)=\begin{pmatrix} y_+\,e^{i(\theta+n\gamma)/2} \\ y_-\,e^{-i(\theta+n\gamma)/2}\end{pmatrix} \tag{3.16}$$

where $x_\pm$ and $y_\pm$ are arbitrary parameters. Now the generalized states of Eq. (2.15),

$$\Psi_n(\boldsymbol{\theta})=B_{n+1,n-1}(\theta_1)\,B_{n+2,n-2}(\theta_2)\cdots B_{n+r,n-r}(\theta_r)\,\Omega_n^{(N)} \tag{3.17}$$

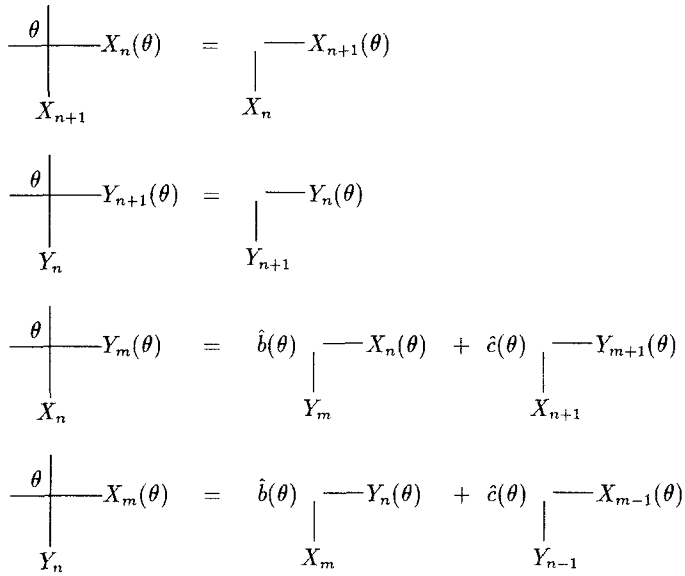

Fig. 3. Graphical representation of the intertwining rules (3.11)–(3.14) in the special case $\theta'=\gamma/2$. Different normalizations of $t_{ab}(\theta)$ with respect to $R(\theta)$ lead to different coefficients in the right-hand sides. Explicitly, $\hat{b}(\theta)=\sin\gamma/\sin(\theta+\gamma/2)$ and $\hat{c}(\theta)=\sin(\theta-\gamma/2)/\sin(\theta+\gamma/2)$. Everywhere X_n is a short-hand notation for $X_n(\gamma/2)$.

have a graphical representation very similar to that of $\Phi(\zeta)$ in Fig. 2: one needs just to make the suitable substitutions of q with r and $|\pm\rangle$ with $X_n(0)$, $Y_n(\theta_l)$, or $\tilde{Y}_n(\theta_l)$ according to Fig. 4 [$\tilde{X}_n$ and $\tilde{Y}_n$ are two-dimensional row vectors defined by

$$\begin{pmatrix}\tilde{Y}_n\\ \tilde{X}_n\end{pmatrix}=(X_n\ \ Y_n)^{-1}$$

see Eq. (2.11)]. One can now repeatedly apply the "intertwining" rules of Fig. 3 to every crossing point of Figs. 2 and 4, starting from the lower right corner. In the first case one arrives in the end at $\Phi(\zeta)$ in the form of a linear combination of states of type $|\alpha_1\cdots\alpha_N\rangle$ with $\alpha_j=\pm 1$ and $\alpha_1+\cdots+\alpha_N=N-2q$, while in the second case one gets $\Psi_n(\boldsymbol{\theta})$ as a linear combination, *with coefficients identical with those of the previous combination, provided* $q=r$ *and* $\zeta=\boldsymbol{\theta}$, of states of type $X_{n_1}(\gamma/2)\otimes\cdots\otimes X_{n_N}(\gamma/2)$. These two combinations can be mapped one into the other by means of the correspondence

$$|\alpha_1\cdots\alpha_N\rangle\Leftrightarrow Z_n^{(\alpha_1)}\otimes Z_{n-\alpha_1}^{(\alpha_2)}\otimes\cdots\otimes Z_{n-\alpha_1-\cdots-\alpha_{N-1}}^{(\alpha_N)} \tag{3.18}$$

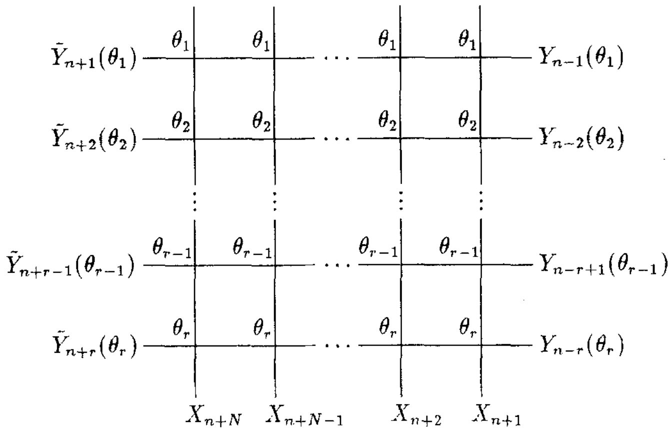

Fig. 4. The "generalized" Bethe states $B_{n+1,n-1}(\theta_1)\cdots B_{n+r,n-r}(\theta_r)\,\Omega_n^{(N)}$. As in Fig. 3, $X_n = X_n(\gamma/2)$.

where $Z_n^{(+)} = X_n(\gamma/2)$ and $Z_n^{(-)} = Y_n(\gamma/2)$. It is not difficult to derive from this an explicit linear map between $\Phi(\boldsymbol{\theta})$ and $\Psi_n(\boldsymbol{\theta})$. To this end, notice first that the matrix $M_n(\theta) = (X_n(\theta)\ \ Y_n(\theta))$ can be written

$$M_n(\theta) = e^{in\gamma\sigma_3/2} M(\theta) \tag{3.19}$$

where $M(\theta)$ is n independent. Then some simple matrix manipulations lead to

$$\Psi_n(\boldsymbol{\theta}) = e^{in\gamma S_3} \mathcal{M} \Phi(\boldsymbol{\theta}) \tag{3.20}$$

where

$$\mathcal{M} = \prod_{k=1}^{N} M(\gamma/2)^{(k)} \exp\left[-i\gamma\sigma_3^{(k)} \sum_{j<k} \sigma_3^{(j)} \right] \tag{3.21}$$

We see from (3.20) that $\Psi_n(\boldsymbol{\theta})$ depends on n only through the factor $e^{in\gamma S_3}$ in front. Therefore the sum over n in Eq. (2.18) can be readily performed to obtain $\Psi_\phi(\boldsymbol{\theta})$ [recall that we have taken $k \to 0$: in the eight-vertex case the coefficients of the intertwining rules, Eqs. (2.9)–(2.12), depend on n, so that $\Psi_n(\boldsymbol{\theta})$ has a much more involved dependence on n]. Taking into account that $\phi = p\gamma$, we see that the sum over n projects $\Psi_n(\boldsymbol{\theta})$ onto its component with $S_3 = p$. Thus, up to an irrelevant factor, we can write

$$\Psi_\phi(\boldsymbol{\theta}) = P(S_3 = p)\, \mathcal{M} \Phi(\boldsymbol{\theta}) \tag{3.22}$$

where we denote by $P(S_3 = p)$ the projector onto the subspace with a definite value p of the third component S_3 of the total spin.

Let us analyze now the arguments $\boldsymbol{\theta}$ in Eq. (3.22). It is clear that such an equation holds for any value of $\boldsymbol{\theta}$, not necessarily a solution of the Bethe Ansatz equations (2.24) or (3.6). Of course, $\Psi_\phi(\boldsymbol{\theta})$ is not an eigenvector of the transfer matrix $\tau(\theta)$ unless $\boldsymbol{\theta}$ satisfies (2.24). Now, when $p = S_3 = 0$, Eqs. (2.24) and (3.6) are identical, since then necessarily $q = r = N/2$. In this particular case the states $\Psi_0(\boldsymbol{\theta})$ and $\Phi(\boldsymbol{\theta})$ are just proportional, provided $\boldsymbol{\theta}$ satisfies (2.24).

When $p \neq 0$ we can distinguish two relevant possibilities:

1. Among the r roots of Eqs. (2.24), $|p|$ have an imaginary part equal to $-\infty$ if $p > 0$ or to $+\infty$ if $p < 0$.
2. The roots "at infinity" are fewer than $|p|$.

In case 1 we find that the real parts of the roots at infinity differ from one another by odd integer multiples of $\pi/2$. In detail, the analysis for $p < 0$ goes as follows. Assuming

$$\theta_k \to +i\infty + u_k, \qquad k = 1, ..., |p| \tag{3.23}$$

with u_k finite, we find that

$$\prod_{k=1}^{|p|} \frac{\sin(\theta_l - \theta_k + \gamma)}{\sin(\theta_l - \theta_k - \gamma)} = e^{-2ip\gamma} \tag{3.24}$$

where the θ_l, $l = |p| + 1, ..., r$ ($r = N/2$: we consider the generic case for which γ is not a rational multiple of π) are finite by hypothesis. The phase factor (3.24) then cancels that in (2.24), leaving for the finite roots the equations

$$\left[\frac{\sin(\zeta_l + \gamma/2)}{\sin(\zeta_l - \gamma/2)}\right]^N = \prod_{k=1}^{q} \frac{\sin(\zeta_l - \zeta_k + \gamma)}{\sin(\zeta_l - \zeta_k - \gamma)} \tag{3.25}$$

where we have quite naturally renamed the finite roots as $\theta_l = \zeta_{l-|p|+1}$ and $q = N/2 - |p|$. Equations (3.25) are completely identical to the conventional Bethe Ansatz equations (3.6). Similarly, the transfer matrix eigenvalue [see Eqs. (2.19) and (2.23)] associated with the roots $\boldsymbol{\theta}$ comprised of $\boldsymbol{\zeta}$ and the roots at infinity (3.23) exactly reduce to that of the conventional Bethe Ansatz [Eq. (3.7)]. To complete the analysis, we have also to consider the equations (2.24) with the infinite roots in the left-hand side. Using (3.23) and

$$u_k - u_{k'} = (\text{odd integer})\,\pi/2, \qquad 1 \leqslant k, k' \leqslant |p| \tag{3.26}$$

one finds that these equations all reduce to the trivial identity $e^{iN\gamma} = e^{iN\gamma}$. This completes the case $p < 0$. The other case, $p > 0$, follows analogously.

We can reformulate this result by saying that any set of roots $(\zeta_1,\ldots,\zeta_q)$ of the conventional Bethe Ansatz equations (3.6) or (3.25) plus $|p| = N/2 - q$ infinite roots fulfilling (3.23) and (3.26) provide an acceptable solution of the generalized Bethe Ansatz equations (2.24). The formula (3.20) relating the eigenstates can then be rewritten, for $S_3 = p > 0$, as

$$\Psi_{p\gamma}(\boldsymbol{\theta}) = P(S_3 = p)\, \mathcal{M}\mathcal{B}^p \Phi(\boldsymbol{\zeta}) \tag{3.27}$$

where

$$\mathcal{B} = \sum_{k=1}^{N} \left[\exp\left(-\frac{i}{2}\gamma \sum_{l<k} \sigma_3^{(l)}\right)\right] \sigma_-^{(k)} \exp\left(\frac{i}{2}\gamma \sum_{l>k} \sigma_3^{(l)}\right) \tag{3.28}$$

is $B(i\infty)$ up to an irrelevant factor. Notice that the overall operator $\mathcal{U}_p = P(S_3 = p)\, \mathcal{M}\mathcal{B}^p$ is independent of the particular set of roots ζ and depends only on p. Hence it provides a true linear endomorphism on the subspace with $S_3 = p$ of the total Hilbert space. Given a "conventional" Bethe Ansatz eigenvector (3.5), (3.6), $\mathcal{U}_p$ maps it into a "generalized" eigenvector of ref. 5. Moreover, since we find the same transfer matrix eigenvalues following both procedures, it is conceivable that the eigenvectors are exactly the same, up to an overall factor, unless the spectrum of $\tau(\theta)$ exhibits "accidental" degeneracies. In other words, $\Phi(\zeta_1,\ldots,\zeta_q)$ is necessarily an eigenstate also of $\mathcal{U}_p$, if $|p| = N/2 - q$, provided the corresponding $\tau(\theta)$ eigenvalue is not degenerate [on the problem of the degeneracies of the transfer matrix almost nothing has been rigorously established, although it appears unlikely that the spectrum of $\tau(\theta)$ is degenerate, for all values of θ, in this anisotropic six-vertex model]. In the Appendix, as an illustrative example, we perform explicitly all calculations in the case $N = 4$, where no degeneracies exist. In this case we find indeed that $\Phi(\zeta_1,\ldots,\zeta_q)$, $q = 0, 1, 2$ is an eigenvector of $\mathcal{U}_{N/2-q}$. The eventual validity of this result for general N would express a new and deep property of the Bethe Ansatz construction.

It should be noticed that the states $\Psi_\phi(\boldsymbol{\theta})$ depend on the arbitrary parameters s and t present in the intertwining vectors (2.8) and (2.22). Equation (3.27) shows that in the six-vertex case such a dependence must be confined to the coefficients of a linear combination of $\tau(\theta)$-degenerate conventional Bethe Ansatz eigenstates. With this new information, perhaps the free parameters s and t might indeed play a crucial role in an eventual solution of the degeneracy problem.

Let us examine now case 2. Clearly this case is possible, i.e., there can be solutions of the generalized Bethe Ansatz equation (2.24) with fewer (than $|p|$) roots at infinity. In a sense the most natural possibility is that no root is at infinity [this is the only relevant case for the "untwisted" conventional equations (3.6)]. One should therefore ask what states correspond to these other solutions. Since both Bethe Ansatz constructions are expected to give all the eigenstates of the transfer matrix, no extra state must correspond to case 2. Indeed, an explicit computation in the case $N=4$ shows that $\Psi_{p\gamma}(\boldsymbol{\theta})$ either identically vanishes, when there are less than $|p|$ roots at infinity, or reproduces states already obtained in case 1. It is likely that this property can be proven in general. We want to stress that the *whole* set of solutions of the "twisted" equations (2.24) corresponds to nonzero and nontrivial Bethe Ansatz states, but for different eigenvalue problems. Let us recall that we have been studying up to now the problem of diagonalizing the transfer matrix *for periodic boundary conditions.* Twisted equations like (2.24) appear, however, also in the different problem with twisted transfer matrix

$$\tau_\phi(\theta) = A(\theta)\, e^{i\phi} + D(\theta)\, e^{-i\phi} \tag{3.29}$$

where now ϕ is an arbitrary twist parameter, not necessarily equal to $p\gamma$. Notice that $\tau_\phi(\theta)$ still belongs to a commuting family and commutes with S_3. In the quantum spin chain language, this six-vertex transfer matrix corresponds to the XXZ Heisenberg chain with twisted boundary conditions

$$\sigma_\pm^{(N+1)} = e^{\pm i\phi}\sigma_\pm^{(1)}, \qquad \sigma_3^{(N+1)} = \sigma_3^{(1)} \tag{3.30}$$

This problem is solved by a "conventional" Bethe Ansatz

$$\Phi_\phi(\boldsymbol{\theta}) = B(\theta_1)\, B(\theta_2) \cdots B(\theta_q)\, |N/2\rangle \tag{3.31}$$

where q must take all values from 0 to $N/2$ and the parameters $\boldsymbol{\theta}$ fulfill the twisted equations (2.24) *and are all finite.* This is just the extremum of case 2. By an analysis very similar to that which led to Eq. (3.27), one could now show that the solutions with $p>0$ roots at infinity correspond to the (generalized Bethe Ansatz) eigenstates of the problem with shifted twist $\phi \to \phi - p\gamma$, yielding the physical interpretation also for all intermediate situations of case 2.

Finally, let us mention also that twisted Bethe Ansatz equations, with running twist $\phi = p\gamma$, $p=0, \pm 1, \pm 2, \ldots$, and $\gamma = \pi/m$, $m = 3, 4, \ldots$, are of fundamental importance in the solution of the RSOS models.[6]

APPENDIX

In this Appendix we shall illustrate in detail, with an explicit calculation for the case of the $N=4$ transfer matrix, the relationship between the two Bethe Ansatz constructions examined in the main body of the paper.

Let us consider first the "conventional" Bethe Ansatz. The 16 states of the four-site system can be divided according to their total S_3 eigenvalue

p. We need consider only the 11 states with $p=0, 1, 2$, the others being obtained simply by overall spin reversal. The state with $p=2$ is just the reference state $|2\rangle = |++++\rangle$. The four vectors with $p=1$ are given by

$$|1, v\rangle = B(\theta_v)\,|2\rangle \tag{A1}$$

where $(\theta_v,\ v=1, 2, 3, 4)$ are the four roots of

$$\left[\frac{\sin(\theta+\gamma/2)}{\sin(\theta-\gamma/2)}\right]^4 = 1$$

Setting $z_v = e^{i\theta_v}$, we find $z_1 = -1$, $z_2 = 1$, $z_3 = (1-\sin\gamma)/\cos\gamma$, and $z_4 = 1/z_3$. Then, introducing the four roots of unity $(\varepsilon_1,..., \varepsilon_4) = (1, -1, -i, i)$, one easily calculates that, up to irrelevant proportionality factors,

$$|1, v\rangle = |+++-\rangle + \varepsilon_v\,|++-+\rangle + \varepsilon_v^2\,|+-++\rangle + \varepsilon_v^3\,|-+++\rangle \tag{A2}$$

In the same way one could find the six $p=0$ eigenstates of the transfer matrix of the form $B(\theta_1)\,B(\theta_2)\,|2\rangle$. It is convenient here to give the arbitrarily twisted generalization of these vectors. Namely, we consider the twisted transfer matrix τ_ϕ of Eq. (3.29) with ϕ arbitrary and $N=4$. The corresponding $S_3=0$ eigenstates $B(\theta_1)\,B(\theta_2)\,|2\rangle$, with θ_1 and θ_2 fulfilling the twisted equations

$$e^{4i\phi}\left[\frac{\sin(\theta_1+\gamma/2)}{\sin(\theta_1-\gamma/2)}\right]^4 = \left[\frac{\sin(\theta_2+\gamma/2)}{\sin(\theta_2-\gamma/2)}\right]^4 = e^{2i\phi}\,\frac{\sin(\theta_1-\theta_2+\gamma)}{\sin(\theta_1-\theta_1-\gamma)} \tag{A3}$$

are proportional to the following explicit expressions:

$$\begin{aligned}
|0, \phi, 1\rangle &= |u_+(\phi)\rangle + \xi_-\;|v_+(\phi)\rangle, & |0, \phi, 2\rangle &= |u_+(\phi)\rangle + \xi_+\;|v_+(\phi)\rangle\\
|0, \phi, 3\rangle &= |u_-(\phi)\rangle + \chi_-\;|v_-(\phi)\rangle, & |0, \phi, 4\rangle &= |u_-(\phi)\rangle + \chi_+\;|v_-(\phi)\rangle\\
|0, \phi, 5\rangle &= |w_+(\phi)\rangle, & |0, \phi, 6\rangle &= |w_-(\phi)\rangle
\end{aligned} \tag{A4}$$

where

$$\begin{aligned}
|u_\pm(\phi)\rangle &= |++--\rangle \pm e^{i\phi}(|+--+\rangle + |-++-\rangle) + e^{2i\phi}\,|--++\rangle\\
|v_\pm(\phi)\rangle &= |+-+-\rangle \pm +e^{i\phi}\,|-+-+\rangle\\
|w_\pm(\phi)\rangle &= |++--\rangle \pm ie^{i\phi}(|+--+\rangle + |-++-\rangle) - e^{2i\phi}\,|--++\rangle\\
\xi_\pm &= \tfrac{1}{2}(e^{i\phi}+1)^{-1}\,\{-\Delta \pm [\Delta^2 + 4(1+\cos\phi)]^{1/2}\}\\
\chi_\pm &= \tfrac{1}{2}(e^{i\phi}-1)^{-1}\,\{\Delta \pm [\Delta^2 + 4(1-\cos\phi)]^{1/2}\}
\end{aligned} \tag{A5}$$

and $\Delta = \cos\gamma$ is Lieb's parameter. We will not give here the explicit form of the corresponding $\tau_\phi(\theta)$ eigenvalues, since these are not needed in the following.

Now we shall examine the generalized Bethe Ansatz. The states with $p=0$ correspond to Bethe Ansatz equations identical to the conventional ones. Hence the generalized eigenstates are obtained from the states (A4), after setting $\phi=0$, by application of the operator $\mathscr{U}_0 = P(S_3=0)\mathscr{M}$ [recall Eqs. (3.22) and (3.27)]. To simplify the formulation, it is convenient to use the symmetric definition (3.16) of the intertwining vectors. Then a direct application of the matrix $\mathscr{M}$ in Eq. (3.21) to the vectors (A4) shows that for $\phi=0$ they are eigenstates also of $\mathscr{U}_0$:

$$\mathscr{U}_0\,|0, 0, v\rangle = \lambda_v\,|0, 0, v\rangle, \qquad v = 1,..., 6$$

with eigenvalues

$$\begin{aligned}
&\lambda_1 = x_1 e^{i\gamma} + y_1 e^{-i\gamma} - z\xi_-, &&\lambda_2 = x_1 e^{i\gamma} + y_1 e^{-i\gamma} - z\xi_+ \\
&\lambda_3 = x_1 e^{i\gamma} + y_1 e^{-i\gamma} + 2z\Delta, &&\lambda_4 = x_1 e^{i\gamma} - y_1 e^{-i\gamma} + 2z\Delta \\
&\lambda_5 = x_1 e^{i\gamma} - y_1 e^{-i\gamma} - 2z\Delta, &&\lambda_6 = x_1 e^{i\gamma} - y_1 e^{-i\gamma}
\end{aligned} \tag{A6}$$

where $x_1 = x_+^2\, y_-^2$, $y_1 = x_-^2\, y_+^2$, and $z = x_+ x_- y_+ y_-$. Next consider the case $p=1$. According to Eq. (3.22), we have to apply $\mathscr{U}_1$ to the four states (A2). Thanks to the small value of N, we can straightforwardly perform this calculation, again finding that these vectors are already eigenstates of $\mathscr{U}_1$. Rather than giving all the details for this case (after all, the transfer matrix for $N=4$ has no accidental degeneracy), it is preferable to perform an explicit check that the extra, *finite* solutions of the twisted equations (A3) do not correspond to new states, when $\phi=\gamma$, under the generalized Bethe Ansatz. To see this, consider the four-by-six matrix $\mathscr{M}_{01} = P(S_3=1)$ $\mathscr{M}\mathscr{P}(S_3=0)$. In the (suitably ordered) standard basis $|\alpha_1\alpha_2\alpha_3\alpha_4\rangle$, $\alpha_j = \pm$, it has the explicit expression

$$\mathscr{M}_{01} = \frac{1}{\eta^2}\begin{pmatrix} x\eta^3 & x\eta^2 & y\eta^3 & x\eta & y\eta^2 & y\eta \\ x\eta^2 & y\eta^3 & x\eta^2 & y\eta^2 & x\eta & y\eta^2 \\ y\eta^3 & x\eta^2 & x\eta & y\eta^3 & y\eta^2 & x\eta \\ y\eta^4 & y\eta^3 & y\eta^2 & x\eta^2 & x\eta & x \end{pmatrix} \tag{A7}$$

where $\eta = e^{i\gamma}$, $x = x_+^2\, y_+ y_-$, and $y = x_+ x_- y_+^2$. Now set $\phi=\gamma$ in expressions (A4) and (A5) and apply (A7). One finds that the six vectors

$|u_\pm(\gamma)\rangle$, $|v_\pm(\gamma)\rangle$, and $|w_\pm(\gamma)\rangle$ exactly reduce to the four states $|1, v\rangle$ of Eq. (A2). In detail, if $s_+ = 1$, $s_- = 2$, $s'_+ = 3$, and $s'_- = 4$, then

$$\begin{aligned} \mathcal{M}_{01}\, |u_\pm(\gamma)\rangle &= (\eta \pm 1)(x \pm y\eta)\, |1, s_\pm\rangle \\ \mathcal{M}_{01}\, |v_\pm(\gamma)\rangle &= (x \pm y\eta)\, |1, s_\pm\rangle \\ \mathcal{M}_{01}\, |w_\pm(\gamma)\rangle &= [x(\eta \mp i) \pm iy\eta(\eta \pm i))]\, |1, s'_\pm\rangle \end{aligned} \tag{A8}$$

Equation (A8) shows that the correct $p = 1$ eigenstates can be recovered from the generalized Bethe Ansatz vectors in both cases: when the right number (in this case just one) of roots of the twisted equations (with $\phi = p\gamma$) are at infinity, and also when fewer, or none at all, are at infinity. In this latter case, however, the correspondence between different sets of roots and independent states is no longer one to one. This fact must be properly taken into account when deriving integral equations describing the system in the thermodynamic limit $N \to \infty$.

To complete the analysis of this $N = 4$ problem, we should consider also the case $p = 2$. This is rather trivial, however, since there is only one state with $p = 2$, the reference state $|++++\rangle$ itself.

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